William Blake, *The Ancient of Days*

PHILOSOPHICAL MATHEMATICS

Infinity, Incompleteness, Irreducibility

A short, illustrated history of ideas

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Dedicated to Maria Clara, age three,
João Bernardo, age six,
and all the children of the world

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Contents

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A special thank you to my wife of sixteen years, the epistemologist, philosopher of science, and Feyerabend specialist, Virginia Maria Fontes Gonçalves Chaitin, who in addition to bearing our two delightful little children, João Bernardo and Maria Clara, has never ceased in her efforts to educate me in her areas of expertise. Our intellectual dialogue is reflected in many pages of this book, not just those where she is explicitly credited for her contributions.

Chapter 1

Setting the Stage

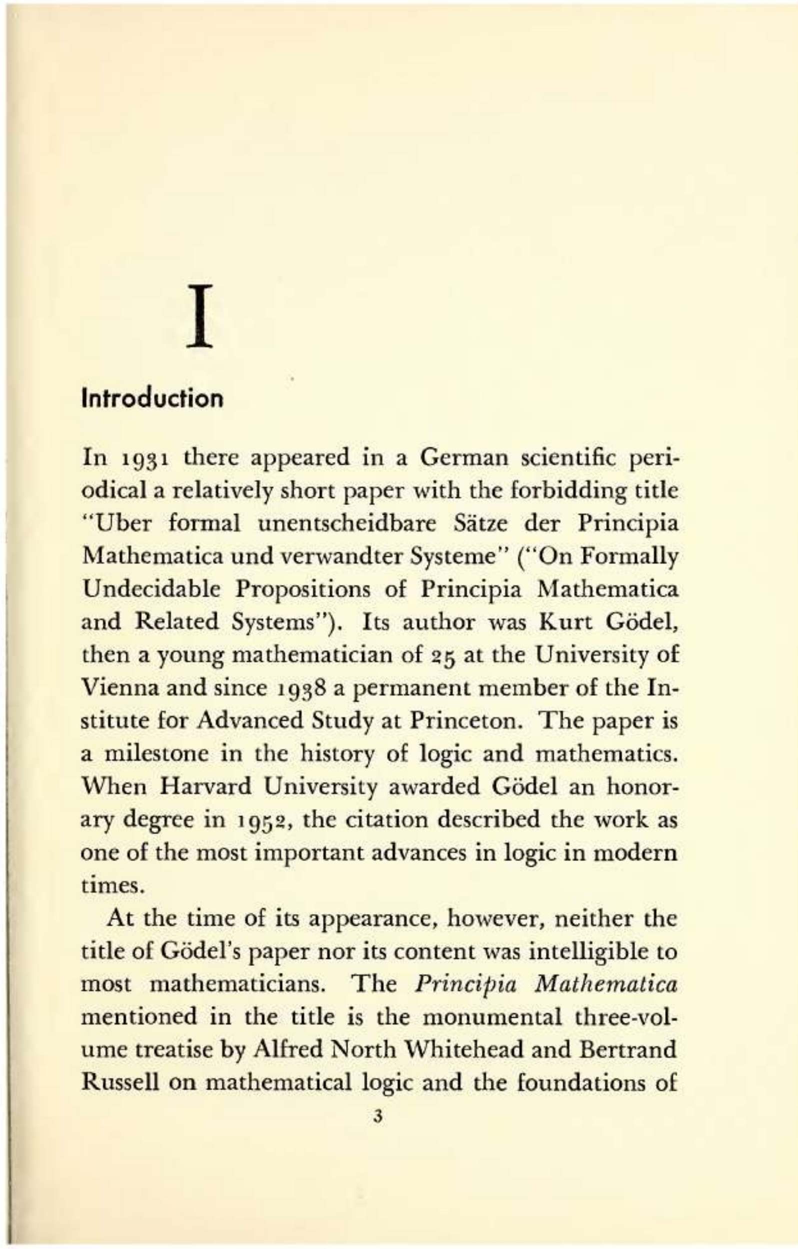
> "I believe in Spinoza's God who reveals himself in the orderly harmony of what exists, not in a God who concerns himself with fates and actions of human beings."—Einstein

The author of this book had the privilege of growing up in post World War II Manhattan, on the East Side a block away from Central Park and the Frick Museum. There he took advantage of the excellent public libraries, where he read many essays written by refugees from Hitler's Europe: Albert Einstein, John von Neumann, Stanislaw Ulam, Hermann Weyl, Max Born, George Polya, Mark Kac, Gian-Carlo Rota... These were physicists and mathematicians of vast culture, including philosophy and the arts. And the essays by Einstein featured frequent references to God, which rubbed off on a impressionable, young and mostly self-taught student.

Thanks to the Cold War, the author also benefited from many special educative programs for bright teenagers interested in science and mathematics. The Americans were scared of the Russians, who had placed the first artificial satellites in orbit. He went to an excellent school called the Bronx High School of Science, and also to a special weekend program for high school students called the Columbia University Science Honors Program.

At Columbia University he took computer programming courses and was able to run computer programs in FORTRAN and symbolic assembly languages on diverse mainframes. Furthermore, he had the run of the Columbia University libraries, including, amazingly enough, access to the stacks where he found many historical volumes including two collected works, the *Opera Omnia* of Leonhard Euler and the *Oeuvres Complètes* of Neils Henrick Abel, great mathematicians both.

It was there in Manhattan in 1958 at the New York Public library, the Donnell branch across the street from the Museum of Modern Art, that an eleven-year old boy encountered a little book that was to send him on a life-long quest: *Gödel's Proof* by Ernest Nagel and James R. Newman. The memorable first page of *Gödel's Proof* is displayed below.



I

Introduction

In 1931 there appeared in a German scientific periodical a relatively short paper with the forbidding title "Uber formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme" ("On Formally Undecidable Propositions of Principia Mathematica and Related Systems"). Its author was Kurt Gödel, then a young mathematician of 25 at the University of Vienna and since 1938 a permanent member of the Institute for Advanced Study at Princeton. The paper is a milestone in the history of logic and mathematics. When Harvard University awarded Gödel an honorary degree in 1952, the citation described the work as one of the most important advances in logic in modern times.

At the time of its appearance, however, neither the title of Gödel's paper nor its content was intelligible to most mathematicians. The *Principia Mathematica* mentioned in the title is the monumental three-volume treatise by Alfred North Whitehead and Bertrand Russell on mathematical logic and the foundations of

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Gödel's Proof, 1958
(Photo courtesy of https://archive.org)

The book you are reading was written by someone who is, of course, now much older than eleven telling about the ideas that inspired him and the landmarks he encountered while on this quest to understand the incompleteness phenomenon discovered by Kurt Gödel, the tip of a very large and mostly submerged iceberg.

The author begs the reader's indulgence for the frequent references to God, perhaps a metaphor, or perhaps more than that, which is for each reader to decide. . .

Let us begin by asking just what makes mathematics so special.

Chapter 2

Why Mathematics?

Quartz Crystals
(Quartz Brésil par Didier Descouens)

Crystals hint at the deep mathematical structure of reality

*** The Platonic World of Mathematical Ideas is Static, Perfect and Eternal ***

According to Plato, the rest is *doxa*, opinion. *Doxa* was contrasted with *episteme* ("knowledge"). Furthermore the sign on the door of Plato's Academy said,

> "Let no one ignorant of geometry enter here."

And remember that the old name for "mathematician" was "geometer."

In math we think like the Gods

Example: There are infinitely many prime numbers

Let us give a beautiful example of the power of pure thought, a proof that there are infinitely many prime numbers. A prime is, of course, a positive integer $\neq 1$ that has no proper divisors, only itself and 1. Our proof is taken from Euclid's Alexandrian compendium of classical Greek mathematics, and it is what is called a *reductio ad absurdum*, a reduction to an absurdity.

Suppose on the contrary that there are only finite many prime numbers. Multiply all of them together and add one, giving us the number M. M isn't exactly divisible by any of those primes because it always leaves the remainder 1. Hence M must itself be a prime, contradiction!

As G. H. Hardy remarks in *A Mathematician's Apology* after presenting this example, in chess one may sacrifice a piece, but in mathematics one offers *the game*.

Euclid's proof is delightfully simple, and much is known about the prime numbers. However some equally simple questions have absorbed lifetimes of fruitless efforts. Here is an example.

Perfect, Deficient and Abundant Numbers

6 is divisible by 1, 2 and 3 which sums to 6, and is therefore *perfect*. 8 is divisible by 1, 2 and 4 which sums to 7, and is therefore *deficient*. And 12 is divisible by 1, 2, 3, 4 and 6 which sums to 16, and is therefore *abundant*.

The next perfect number is 28 which is divisible by 1, 2, 4, 7 and 14 which sums to 28.

Are there infinitely many even perfect numbers? Are there any odd perfect numbers? Amazingly enough, nobody seems to know. Our current mathematical tools cannot answer such questions.

There are also what are called *amicable pairs*, in which each number is the sum of the proper divisors of the other. 220 and 284 are an example of an amicable pair. And perhaps there are even amicable rings. Euler, whom we shall talk about in Chapter 10, was good at finding examples of such things.

This is Plato's static world of hard, perfect, eternal truths, of such concern to ancient Greek intellectuals. It is not an easy world to navigate in. But to them and even to some modern mathematicians it has a cold, austere, inhuman, and other-worldly beauty, illuminated by the tremendously intense, even savage, light of **absolute truth**. Like the blinding, lethal light of the great god Apollo as he revealed himself to one of his mortal lovers, at her very much mistaken request. Or like the god Krishna in the Bhagavad Gita, whom the archer Arjuna begs to stop gradually assuming his divine form because he was already "brighter than a thousand suns."

Ontology: Is the world built out of mathematics? Is God a mathematician?

To be continued in Chapter 11.

Literature

1. G. H. Hardy, *A Mathematician's Apology*, Cambridge University Press, 1940, *With a foreword by C. P. Snow*, 1967. [A portrait of a creative artist.]

2. E. T. Bell, *Men of Mathematics: The Lives and Achievements of the Great Mathematicians from Zeno to Poincaré*, Simon & Schuster, 1937. [Has inspired many people to become mathematicians.]

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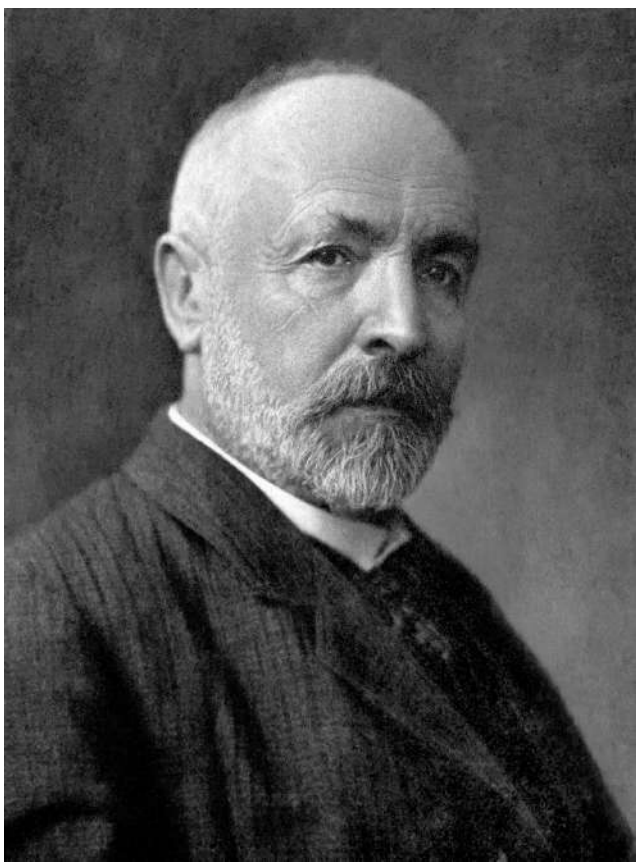
Chapter 3

As if Summoned from the Void: Cantor's Theory of Infinite Sets, 1895, 1897



Georg Cantor

In one of his dialogues, Galileo notes that the positive integers and their squares can be put into a one-to-one correspondence:

$$1 \leftrightarrow 1,\ 2 \leftrightarrow 4,\ 3 \leftrightarrow 9,\ 4 \leftrightarrow 16,\ 5 \leftrightarrow 25, \ldots$$

But the squares constitute only a small fraction of the positive integers. Therefore, Galileo says, there is no way to compare the sizes of infinite sets.

Actually the exact opposite is the case. The central idea of Cantor's theory of infinite sets is that two infinite sets have exactly the same cardinality, the same size, the same number of elements, if there is a one-to-one correspondence between the elements of both sets leaving nothing out.

And the second central idea of Cantor's theory is that the set of all the subsets of an infinite set is always more numerous, is a much bigger infinite set, than the original set.

So start with the set of all positive integers $\{1, 2, 3, \ldots\}$, take all the subsets of that, then all the subsets of the resulting set, and so forth and so on, obtaining an infinite sequence of bigger and bigger infinite sets. Then take the union of all of them, producing an infinite set that is a whole lot bigger, and continue like this, onwards and upwards, yearning for God, but never attaining Him.

Cantor's ordinal and cardinal numbers

Furthermore, Cantor's breathtaking conception includes two transfinite sequences of new kinds of numbers, his ordinals and his cardinals. Here they are.

There is the series of ordinal numbers

$$0, 1, 2, \ldots \omega, \omega+1, \omega+2, \ldots 2\omega, 2\omega+1, \ldots \omega^2 \ldots 6\omega^2+3\omega+7 \ldots \omega^3 \ldots \omega^\omega \ldots \omega^{\omega^\omega} \ldots$$

until ϵ_0,

$$\epsilon_0 = \omega^{\omega^{\omega^{\omega^{\cdots}}}},$$

the smallest solution of

$$x = \omega^x,$$

and beyond.[1]

And there is a corresponding series of cardinal numbers

$$\aleph_0, \aleph_1, \aleph_2, \ldots, \aleph_\omega \ldots \aleph_{\omega^\omega} \ldots \aleph_{\epsilon_0} \ldots,$$

one $\aleph_\alpha$ for each ordinal number α.

Now let's put these new numbers to a good use.

The set of all positive integers $\{1, 2, 3, \ldots\}$ has cardinality $\aleph_0$. The set of all its subsets has cardinality $\aleph_1$. Continuing like this, we drive past $\aleph_2, \aleph_3 \ldots$ until we take the union of this infinite sequence of sets of subsets, which has cardinality $\aleph_\omega$, and so forth and so on, upwards and onwards, endlessly.

Von Neumann ordinals

Each ordinal can be thought of as the set of all smaller ordinals.[2] Thus 0

$$0 = \{\ \}$$

[1] Here we are using non-standard notation. I believe that 2ω looks nicer than $\omega 2$, and for our purposes it makes no difference.

[2] This beautiful idea is not due to Cantor. It was a later contribution by John von Neumann, whom we shall study in Chapter 7.

is the empty set, 1 is

$$1 = \{0\} = \{\{\ \}\},$$

2 is

$$2 = \{0, 1\} = \{\{\ \},\ \{\{\ \}\}\},$$

3 is

$$3 = \{0, 1, 2\} = \{\{\ \},\ \{\{\ \}\},\ \{\{\ \},\ \{\{\ \}\}\}\}.$$

And

$$\omega = \{0, 1, 2, \ldots\},$$
$$\omega + 1 = \{0, 1, 2, \ldots, \omega\} = \{0, 1, 2, \ldots, \{0, 1, 2, \ldots\}\},$$
$$\omega + 2 = \{0, 1, 2, \ldots, \omega, \omega + 1\},$$
$$2\omega = \{0, 1, 2, \ldots, \omega, \omega + 1, \omega + 2 \ldots\}.$$

And so forth and so on, forever and ever.

Cantor's Diagonal Argument, 1895

I suppose that I must prove that the set of all the subsets of a set is bigger than the original set, infinitely bigger. Let's consider a special case that perfectly illustrates the general idea, which is how does one prove that the set of all subsets of $\{1, 2, 3, \ldots\}$ is a bigger infinite set than it is?

Following in Euclid's footsteps, we employ a *reductio ad absurdum*.

Let's suppose on the contrary that there is a one-to-one correspondence between the positive integers and all its subsets, and ask yourself whether the subset that corresponds to n contains n or not. We construct a so-called diagonal set that contains n if and only if the subset that corresponds to n doesn't contain n.

This diagonal set must be a new subset of $\{1, 2, 3, \ldots\}$, one that was not in our original list. Think about it!

Therefore there can be no one-to-one correspondence between a set and the set of all its subsets. *C'est fini!*

The paradoxes of set theory

The diagonal argument provides a lovely proof that the set of all the subsets of a set is bigger than the original set. But apply this argument to the universal set, the set of everything, and you get something bigger than everything, an impossibility.

Similarly, recall that von Neumann defines each ordinal as the set of all smaller ordinals. Then the set of all ordinals Z is the biggest ordinal, but $Z + 1$ is an even bigger ordinal! This is called the Burali-Forti paradox.

Cantor was not unduly perturbed by these paradoxes, because he was doing mathematical theology, and it is inherently paradoxical for a finite mind to attempt to understand God, who is totally and completely infinite, in fact, beyond comprehension.

An irresolvable paradox may be okay in theology but it most certainly is not welcome in pure mathematics. Here are two reactions to Cantor's theory by famous mathematicians:

> *David Hilbert*: "No one shall expel us from the paradise discovered by Cantor."

> *Henri Poincaré*: "Set theory is a disease from which future generations will be happy to have recovered."

The Zermelo-Fraenkel axiomatic formulation of set theory was designed to avoid the paradoxes. But in my opinion it obscures Cantor's essential contribution, which was his discovery of the open-endedness of mathematical imagination. Cantor's theory in its original, unadulterated form argues against a static, perfect, eternal view of mathematical truth and in favor of the idea that

*** The Platonic World of Mathematical Ideas is *In Statu Nascendi* ***

Anyway, as we shall see in the next chapter, Hilbert then set out to put math on a firm footing, to propose a way, he hoped, to eliminate paradoxes and restore absolute certainty.

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Chapter 4

David Hilbert's Metamathematics, c. 1900

David Hilbert

The 1900 International Congress of Mathematicians in Paris

In 1900 Hilbert led the German delegation to Paris and delivered a stirring address, seeking to set the tone for the mathematics of the new century. Here are two excerpts:

> Who of us would not be glad to lift the veil behind which the future lies hidden; to cast a glance at the next advances of our science and at the secrets of its development during future centuries? What particular goals will there be toward which the leading mathematical spirits of coming generations will strive? What new methods and new facts in the wide and rich field of mathematical thought will the new centuries disclose?
>
> This conviction of the solvability of every mathematical problem is a powerful incentive to the worker. We hear within us the perpetual call: There is the problem. Seek its solution. You can find it by pure reason, for in mathematics there is no *ignorabimus*.

This memorable speech remains the most forceful expression of the mathematical spirit that has ever been uttered. It is full of Hilbert's energy, enthusiasm and optimism, the energy, enthusiasm and optimism of a great mathematician.

But there was a problem lurking in the background: the paradoxes of set theory, other paradoxes, and controversies over the correct methods to employ in mathematical proofs. Hilbert set out to restore stability and order to the world of mathematics, by proposing what came to be known as the Hilbert program.

Hilbert's Program to Restore Order and Stability to Mathematics

Here was Hilbert's proposal:

First of all, using mathematical logic, formulate a formal axiomatic theory for all of mathematics, a TOE, Theory of Everything. This would employ an artificial language with perfect syntax to eliminate ambiguities, and there should also be a mechanical procedure to check if a proof satisfies the rules or not.

Second, convince the entire mathematical community to unite behind this TOE.

Result: absolute certainty would be restored!, a most laudable goal. Mathematical truth would become completely objective, not subjective. It would be black or white, never gray.

However, it was never Hilbert's intention to force mathematicians to actually use this TOE in their everyday research. Hilbert's goals were philosophical, not practical. In practice, mathematicians would continue as before, writing proofs in German or French...

Metamathematics

Hilbert's program for rescuing mathematics also creates a new field of mathematics, called metamathematics, that studies formal axiomatic theories from the outside, from above, employing mathematical methods, a psychiatric self-analysis, as it were.

Let us begin doing metamathematics.

The first step is to notice that one can enumerate all the theorems of a formal axiomatic theory. You can run through the tree of all possible deductions from the axioms, or you can apply the proof-checker to all possible strings of characters selected from the alphabet of the formal axiomatic theory. Either way, you get, very slowly, all the theorems that can be demonstrated, in order of the size of the proofs, a process that never comes to an end.[1]

Now suppose that you, a real life mathematician, are interested in determining whether a certain assertion A is true or false. Well, if you are very, very patient, in principle at least, you can just start enumerating all the theorems in your TOE until you find a proof of A, or you find a proof of not A. That's it, you're done, absolutely brainlessly, absolutely mechanically, at least in principle.

How likely do you think it is that this is possible? Not very. Perhaps it is preferable for there not to be such a TOE, otherwise mathematicians will all be out of a job!

Of course, in practice this procedure cannot be carried out, it would be incredibly slow. But in theory, such a TOE would trivialize mathematics, which may make you suspect that it is not possible, not even in theory.

The great mathematician Henri Poincaré, who believed in intuition, not logic, mercilessly criticized Hilbert's formal axiomatic theories, comparing them to a gigantic sausage machine. He missed the point, for the Hilbert program led to the creation of digital computing machinery. Indeed, two mathematicians, John von Neumann and Alan Turing, both inspired by Hilbert, respectively helped to create the computer industry in the U.S. and in the U.K.. But Poincaré was not entirely mistaken, for the quest for absolute truth, for absolute certainty, proved remarkably elusive, as we shall see in the next chapter, on Kurt Gödel.

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[1] This idea was later to play an important role in the work of Alan Turing and even more so in the thinking of Emil Post.

Chapter 5

Kurt Gödel's Incompleteness Theorem, 1931

Kurt Gödel

"Wir müssen wissen, wir werden wissen!"—David Hilbert, 1930
We must know, we will know!

This was Hilbert's rousing rallying cry for mathematicians, in what was practically his retirement address at a meeting for mathematicians. Meanwhile, in a satellite event, the young Kurt Gödel was announcing his incompleteness theorem to an uncomprehending audience. Only John von Neumann instantly understood.

Von Neumann could have proved Gödel's theorem himself, but it had never occurred to him that Hilbert's program could be flawed. Henceforth von Neumann was to support Gödel in every possible way.

Gödel, like the poet William Blake, was an unworldly figure.[1] But in his own world, the world of ideas, he was daring, imaginative, and extremely competent. In this world, however, he was far from being as sophisticated as von Neumann, a banker's son.

It was von Neumann who managed to rescue Gödel and his wife from wartime Vienna. Von Neumann obtained a still neutral United States visa for them, whereupon they traveled around the world to avoid the war zone and reach Princeton, New Jersey.

Later it was von Neumann who managed to get Gödel a permanent appointment at the Institute for Advanced Study, finally putting an end to years of short term appointments that had to be periodically renewed and that made Gödel nervous. This boon proved to be a mixed blessing, however, as Gödel subsequently felt less pressure to publish and could devote himself exclusively to studying Leibniz.

His friend Karl Menger, during an IAS visit, encouraged Gödel to work on his own ideas instead of studying Leibniz's. But Gödel could not be dissuaded. In response to Menger's entreaties, he revealed his suspicions that some very important discoveries by Leibniz had been suppressed by "forces inimical to human progress." Menger was naturally skeptical, but was surprised by the considerable bibliographic evidence of anomalies that Gödel had managed to assemble from the excellent Princeton University libraries.

Rebecca Goldstein in her biography of Gödel tells a story, perhaps apocryphal, but nevertheless worth repeating. In reply to a boring dinner companion who had described at length his astrophysical discoveries, an annoyed Gödel snapped, "I don't believe in empirical science, I only believe in *a priori* truths!"

In spite of the kantian terminology, this is platonism, pure platonism. Nevertheless Gödel and Einstein were close friends. Einstein, an empiricist, and Gödel, a platonist, were often seen walking together through the streets of Princeton, intensely conversing in German. But perhaps some of Einstein's beliefs did rub off on Gödel, though transposed into platonist garb.

Here is a very important statement by Gödel that has a distinctly quasi-empirical air.[2]

> "Furthermore, however, even disregarding the intrinsic necessity of some new axiom, and even in case it had no intrinsic necessity at all, a decision about its truth is possible also in another way, namely, inductively by studying its "success," that is, its fruitfulness in consequences and in particular in "verifiable" consequences, i.e., consequences demonstrable without the new axiom, whose proofs by means of the new axiom, however, are considerably simpler and easier to discover, and make it possible to condense into one proof many different proofs. The axioms for the system of real numbers, rejected by the intuitionists, have in this sense been verified to some extent owing to the fact that analytical number theory frequently allows us to prove number theoretical theorems which can subsequently be verified by elementary methods. A much higher degree of verification than that, however, is conceivable. There might exist axioms so abundant in their verifiable consequences, shedding so much light upon a whole discipline, and furnishing

[1]But not as unworldly as is usually thought. In his youth he had frequented night clubs in Vienna, and at Princeton he would take the trouble to travel to New York City for musical comedies.

[2]Please see the end of Chapter 8 for more on quasi-empiricism in mathematics.

> such powerful methods for solving given problems (and even solving them, as far as that is possible, in a constructivistic way) that quite irrespective of their intrinsic necessity they would have to be assumed at least in the same sense as any well established physical theory."—Gödel, "What is Cantor's Continuum Problem?," 1947

This is drawn from a discussion of the need for new axioms in set theory, a subject that is much more intimidating than the relatively pedestrian version that I presented in the chapter on Cantor. In fact, set theory is a thorny, challenging subject for only the most brilliant, most adventurous mathematical minds.

Just to reinforce the message, here is Gödel's other fundamental text on quasi-empiricism, from a 1944 *festschrift* in honor of Bertrand Russell.

> "The analogy between mathematics and a natural science is enlarged upon by Russell also in another respect (in one of his earlier writings). He compares the axioms of logic and mathematics with the laws of nature and logical evidence with sense perception, so that the axioms need not necessarily be evident in themselves, but rather their justification lies (exactly as in physics) in the fact that they make it possible for these "sense perceptions" to be deduced; which of course would not exclude that they also have a kind of intrinsic plausibility similar to that in physics. I think that (provided "evidence" is understood in a sufficiently strict sense) this view has been largely justified by subsequent developments, and it is to be expected that it will be still more so in the future. It has turned out that (under the assumption that modern mathematics is consistent) the solution of certain arithmetical problems requires the use of assumptions essentially transcending arithmetic, i.e., the domain of the kind of elementary indisputable evidence that may be most fittingly compared with sense perception. Furthermore it seems likely that for deciding certain questions of abstract set theory and even for certain related questions of the theory of real numbers new axioms based on some hitherto unknown idea will be necessary. Perhaps also the apparently unsurmountable difficulties which some other mathematical problems have been presenting for many years are due to the fact that the necessary axioms have not yet been found. Of course, under these circumstances mathematics may lose a good deal of its "absolute certainty;" but, under the influence of the modern criticism of the foundations, this has already happened to a large extent. There is some resemblance between this conception of Russell and Hilbert's "supplementing the data of mathematical intuition" by such axioms as, e.g., the law of excluded middle which are not given by intuition according to Hilbert's view; the borderline however between data and assumptions would seem to lie in different places according to whether we follow Hilbert or Russell."—Gödel, "Russell's Mathematical Logic," 1944

In line with these remarks by Gödel, the set theory community has in fact in recent years decided to add a powerful new axiom, *Projective Determinacy*, that is not at all self-evident.

After this, it may seem like an anticlimax to turn to Gödel's proof of his incompleteness theorem. The basic idea is extremely simple.

"This statement is unprovable!" is true if and only if it is unprovable!

The technical complications reside in expressing this as an arithmetical statement within what is called Peano arithmetic, the standard formal axiomatic formulation of the non-negative integers with addition, multiplication and equality.

At any rate, the consequences for the Hilbert program are devastating.

Therefore there is no TOE for pure mathematics.

Therefore mathematics does not provide absolute certainty.

As we ponder this unexpected turn of affairs, perhaps a few criticisms are nevertheless in order.

Gödel's proof is much too complicated and much too clever. When you find the correct, the natural context for thinking about a problem, the proof becomes almost obvious. As an example of this, Alexandre Grothendieck neglected to publish a proof of an important theorem because it was *une astuce*, a trick. On the other hand, according to J. E. Littlewood, *A Mathematician's Miscellany,* a mathematician is known for the number of bad proofs that he or she has published, because pioneering work is difficult.

Gödel's proof also leaves open the question of

How often are mathematical truths unprovable?

And does incompleteness apply to less artificial questions, to problems that mathematicians really care about? Yes, there is in fact one, only one, good example of this:

Only non-artificial example: Cantor's continuum hypothesis is independent of abstract axiomatic set theory (Kurt Gödel, Paul Cohen)

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Chapter 6

Alan Turing's Halting Problem, 1936

Alan Turing

General, Mechanical procedure = algorithm

Turing's 1936 paper features the idea of a universal Turing machine, which can be programmed to carry out any algorithm. It thus, in a sense, anticipates modern computer technology, at least as a mathematical idea, one that still would require a great deal of engineering ingenuity to actually implement.

But this was not at all Turing's original goal. His goal was to advance Hilbert's metamathematical research program and to provide an entirely different proof of incompleteness, totally unlike the one in Gödel's 1931 paper. He did this by asking the following seemingly trivial but actually extremely profound question:

Is there a general procedure for determining whether a self-contained computer program goes on forever or whether it eventually stops?

Assuming the computer has unlimited storage and time, of course. This is a conceptual problem, not a practical one.

No! Proof: Apply Cantor's diagonal argument to the computable real numbers

In a nutshell, the idea of Turing's proof of the undecidability of the halting problem is as follows: Whether the Nth computer program ever outputs an Nth digit cannot be mechanically decided, because otherwise you could compute an uncomputable real number. Instead of explaining this in more detail, I prefer to give a more modern proof.

A more modern program-size proof

We anticipate here the study of program-size complexity in Chapters 8 and 9. Fix the computer programming language, and measure the size of programs in characters of code.

Assume on the contrary that there is an algorithm to decide whether or not a self-contained computer program ever halts. Given the positive integer N, we apply this algorithm to all programs up to N characters in size, and find all of the programs that halt. Then we run all of them until they halt, and combine all of their outputs into one big output. The program to do all of this can be written in the same programming language, and it is only $\log_{10} N + c$ characters long. But

$$\log_{10} N + c \ < \ N$$

for all sufficiently large N, a contradiction, because our $\log_{10} N + c$ characters long program is too small to be able to produce the immense output that it does.

Very well, the halting problem is algorithmically unsolvable. But how does this furnish us with a deeper proof of incompleteness? Well, all you have to do is to realize that

The Set of All Theorems Provable in a Formal Axiomatic Theory Can be Mechanically Enumerated (Very, Very Slowly)

As Turing notes in his 1936 paper, given any formal axiomatic theory T *à la* Hilbert, there is always an algorithm to run through all possible proofs and find all the theorems, an endless computation, completely mechanically.

Therefore Uncomputability implies Incompleteness

Because, if we assume that T has the property that you can only prove that a program p halts or that p fails to halt if this is actually the case, and we also assume that T always enables one to decide, then we could obtain an algorithm to resolve the halting problem by running through all possible proofs in T until we either find a proof that p halts or a proof that p never halts, which is impossible, because no such algorithm can exist.

Voilà, a new proof of incompleteness! But it is a little strange. Gödel constructs a true arithmetical assertion that is unprovable within Peano arithmetic, but who ever heard of the halting problem? Nobody before 1936. Fortunately, it is possible to convert the halting problem into a question about diophantine equations, which go back to Diophantus of Alexandria.

Hilbert's 10th Problem: arithmetic versions of the halting problem

A diophantine equation is one that has integer constant coefficients, positive integer constant exponents, and one or more variables that are non-negative integers. It is built up using only multiplications and additions and subtractions, in any which way, but algebraic expressions are often collected into a polynomial such as $3x^2 + 7x + 2$.

Here is an example of a diophantine equation:

$$x^2 + y^2 = z^2$$

has the solution $x = 3, y = 4, z = 5$ because

$$3^2 + 4^2 = 9 + 16 = 25 = 5^2.$$

In contrast, the equation

$$x^3 + y^3 = z^3$$

has no non-negative integer solutions, except for trivial ones with $xyz = 0$. This is a special case of Pierre de Fermat's famous last theorem, whose original proof has been lost, but which was recently demonstrated in full generality by Andrew Wiles using techniques that Fermat could not possibly have employed.

Diophantine equations have been studied since Diophantus of Alexandria two millennia ago. They have a long history, and a high pedigree in the world of pure mathematics. In contrast the halting problem dates from 1936, and feels rather like a new boy at school struggling to be accepted by his classmates. Fortunately it turns out that there is an intimate connection between the halting problem and diophantine equations.

It turns out that the halting problem is equivalent to asking whether or not a diophantine equation has a solution. It is not just about computer programs. This was how it was shown that there is no algorithm to determine whether or nor a diophantine equation has a solution, which was the tenth problem in Hilbert's famous list of 23 challenges for the new century that he presented at the International Congress of Mathematicians in Paris in the year 1900. Hilbert did not anticipate a negative solution; he asked for an algorithm to determine whether or not a diophantine equation can be solved.

This equivalence between solving the halting problem and determining whether a diophantine equation can be solved was demonstrated by Yuri Matijasevic based on previous work by Julia Robinson, Hilary Putnam and Martin Davis.

Matijasevic constructed a diophantine equation with the parameter k and many additional variables which has an infinite number of solutions if the kth computer program halts and which has no solution if the kth computer program never halts.

The missing piece of the puzzle that had eluded Julia Robinson, Hilary Putnam and Martin Davis turned out to be a simple property of the Fibonacci numbers. Matijasevic was 22 at the time, as Newton described his similar experience, "in the prime of my age for invention."

Turing Oracles, 1938

But the halting problem was not Turing's only significant contribution to pure mathematics and to mathematical philosophy. In 1938, he came up with another important idea.

Turing's magnificent conception in his 1938 paper on oracles is of a transfinite hierarchy of increasingly powerful oracles O_α, one for each constructive ordinal α. I shall not attempt to define a constructive ordinal here precisely; roughly speaking, it is one that has a name and can reached in the limit from below via a computable fundamental sequence. Here are some examples of such sequences:

$$\omega, \omega+1, \omega+2, \ldots \to 2\omega$$

$$\omega, 2\omega, 3\omega, \ldots \to \omega^2$$

$$\omega, \omega^2, \omega^3, \ldots \to \omega^\omega$$

$$\omega, \omega^\omega, \omega^{\omega^\omega}, \ldots \to \epsilon_0$$

These are examples of limit ordinals. Successor ordinals like $\alpha+1$ are constructive if α is.

Anyway, it is intuitively clear that the ordinals through ϵ_0 are constructive ordinals, and furthermore they have rather convenient names.[1]

[1]Afterwards things become more complicated, because one has to keep inventing new names, new notations, forever and ever, an inherently creative, open-ended task... This is explained rather well in the book by John Stillwell in the literature section of Chapter 3 on Cantor.

Recall that each ordinal β may be conceived as the set of all ordinals α less than β. Similarly, the oracle O_β works as follows. A computer program p can give the oracle O_β two pieces of information:

1. The name of an ordinal α that is less than β, and

2. the program p' that will be given access to the oracle O_α.

The oracle O_β then solves the halting problem for p', it tells p whether or not p' halts when run with access to the oracle O_α. p can then proceed to use this information about p' in any way that it cares to.

Thus Turing oracles O_β become increasingly powerful as β increases, because they can solve the halting problem for all programs having access to less powerful oracles O_α with $\alpha < \beta$.

Turing on Machine Intelligence

After World War II, Turing's life gradually began to crumble. But he was able to produce yet another classic, a memorable publication on artificial intelligence. Most unfortunately the U.K. did not take advantage of Turing's genius. Indeed, they persecuted him.

Nevertheless, in 1950 Turing published "Computing Machinery and Intelligence" in the journal *Mind*, which has been an inspiration for all of us who wonder if machines will replace human beings. Hopefully not! But Turing clearly thought, and perhaps even hoped, that this might be the case. Considering what he went through, it is understandable.

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Chapter 7

John von Neumann on Natural and Artificial Software, 1948, 1951

John von Neumann
(Photo courtesy of Los Alamos National Laboratory.)

Turing's Work on Morphogenesis was a mistake

At the end of his troubled life, Turing used nonlinear partial differential equations to generate biological patterns, but *evo-devo* (evolutionary developmental biology) teaches us that pattern generation is actually algorithmic via DNA software.

Von Neumann realized that Turing's work was fundamental to biology, but Turing himself failed to do so.

We have already encountered von Neumann twice, in the beautiful von Neumann ordinals, and in the story of how he would help Gödel whenever he could. Let me tell you more about this extraordinary mind.

A child prodigy, von Neumann was a polymath who worked in many different areas. I loved his book, written with Oskar Morgenstern, *Theory of Games and Economic Behavior*. And in *Mathematical Foundations of Quantum Mechanics*, von Neumann provided the Hilbert space formulation for quantum mechanics, named in honor of his teacher, David Hilbert.

In fact, it seemed to me as a young student that von Neumann was perfectly capable of creating a new field of mathematics every morning before breakfast. I was determined to give it a shot myself!

Von Neumann was instrumental in creating the computer industry in the U.S. via a series of extremely influential reports coauthored with Herman Goldstein and Arthur Burks. He also had a computer built in the basement of the Princeton Institute for Advanced Study, used for secret hydrogen bomb detonation calculations, and widely copied (the so-called von Neumann architecture). Turing was equally influential in the U.K., first with a special-purpose computer for cryptography, then with a general-purpose computer, MADAM, the Manchester Automatic Digital Machine.

Furthermore, in a remarkable 1948 talk and 1951 paper, von Neumann predicts DNA, the software of life, an amazing mathematical prophecy. In an Imre Lakatos "rational reconstruction" this little-known paper would have jump-started molecular biology.

In the event, the young physicists who, disgusted by nuclear weapons, abandoned physics to study viruses, the hydrogen atom of biology, were inspired by Erwin Schrödinger's book *What Is Life?*, not by von Neumann's more perceptive paper that realized that the essential new idea needed to understand biology was the idea of software, as presented for the first time in Turing's seminal 1936 paper. Turing himself did not realize this.

Software is everywhere in biology, and the human genome can now be sequenced in one hour! In meta-genomics, you extract DNA fragments from a drop of pond water, you piece them together using neural net technology, and you assemble a library of genomes, a complete ecology, from one drop of water!

Let us review von Neumann's paper more carefully.

First of all, in it he is courageous enough to refer to *artificial automata*, computers, and to *natural automata*, biological organisms. In both cases according to von Neumann, the fundamental mathematical idea is that of software.

It is software that provides the plasticity of the biosphere and of computer technology.

Nature invented software eons before humanity did. Evo-devo is software archeology.

After this spectacular beginning, von Neumann abandoned biology for a while, due to consulting for the military and his work in the Atomic Energy Commission in Washington, in my opinion a complete waste of his remarkable talents, but no one who hasn't lived through that period should presume to judge.

He did finally return to biology.

I view his complicated cellular automata self-reproducing computer project as a giant crossword puzzle he worked on as a respite from his Washington duties, but not really at von Neumann's intellectual level.

Why do I say this?

Well, because the fundamental principle in biology resides in evolution, not self-reproduction. Most animals reproduce sexually and therefore neither the father nor the mother is copied exactly, their children are different.

In talk transcripts, and notes that were eventually published posthumously, it is clear that von Neumann understood this perfectly.

But this giant crossword puzzle was a welcome respite from his soul-searing Cold War duties.

The next step: Metabiology

What could von Neumann have done if he had not been struck down prematurely due to cancer provoked by the radiation damage he sustained at Los Alamos during the war? By the way, the great physicist Enrico Fermi also perished at about the same time for the same reason.

Well, biological evolution is a million-pound marshmello. One has to simplify in order to be able to prove any theorems, and the job of the mathematician is to prove theorems.

The key insight, in my opinion, is to study the random evolution of computer software instead of the random evolution of actual biological software, DNA. So let's make the model as simple as possible. Our organisms will be software organisms, without bodies and without metabolism. And to make things even simpler, we consider only a single mutating organism at a time, not a population.

That reduces the problem of evolution to a classical problem in probability theory and statistical physics, a random walk. To be more precise, we consider a *hill-climbing random walk in software space*, a new kind of mathematical object. The random walk is called hill-climbing, because only mutations that increase the fitness are accepted. Otherwise our software organism remains the same.

And what is our measure of the fitness of a software organism? The simplest possible choice is to appeal to the Busy Beaver problem, a version of the halting problem, and to have each organism calculate a single positive integer. The bigger the integer, the better the organism.[1]

Finally, we have to specify our mutational model. Easy, choose a program at random, an N-bit self-delimiting program with probability 2^{-N}, the measure space employed in our chapter on the halting probability Ω. Give this randomly chosen program the current organism as input, and let it produce the new mutated organism as output.

So, our model employs global algorithmic mutations, not indels and snips.

One annoying detail with this elegant model is that it employs Turing oracles, because a software organism may never halt, and because an algorithmic mutation may fail to generate a new organism. One has to be a little bit careful about restricting the use of the oracle...

But it turns out, as my wife, epistemologist and philosopher of science Virginia Chaitin, points out, that these oracles correspond to the environment in Darwinian

[1] Another interesting possibility would be for each software organism to calculate a single constructive ordinal, the bigger, the better.

evolution, because the new information, the biological creativity as it were, comes from them. So the oracles are actually an asset, not a liability.

I call this metabiology, and my book *Proving Darwin: Making Biology Mathematical* is all about this model.[2] In honor of von Neumann, from whom I learned so much, the book begins with a portrait of von Neumann and ends with an extensive extract from his 1948/1951 paper on biology that we have been discussing in this chapter.

I wish I could have shown it to von Neumann. He should have done it himself.[3]

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[2] The title is an exaggeration because this was a trade book, not an academic book. Commercial publishers want to sell!

[3] For more on metabiology, please see the chapter on creativity in mathematics and in biology.

Chapter 8

Weyl, Leibniz and the Problem of the Elegant Program, c. 1965, 1974

"As God cogitates and calculates, so the world is made."—Leibniz

Hermann Weyl

G. W. Leibniz

Hermann Weyl, perhaps David Hilbert's best student, in addition to his books on mathematics and mathematical physics, also published two books on philosophy and one on aesthetics. Here are some of the titles: *Philosophy of Mathematics and Natural Science; The Open World: Three Lectures on the Metaphysical Implications of Science; Symmetry; Space, Time, Matter; The Theory of Groups and Quantum Mechanics.*

It was Weyl who discovered an extremely important idea buried in parts V and VI of Leibniz's *Discours de métaphysique* (1686): **If arbitrarily complex laws are permitted then the concept of law becomes vacuous because there is always a law!** A relevant excerpt is displayed below.

> *mouvement reglé*. Mais quand une regle est fort composée, ce qui luy est conforme, passe pour irregulier.
>
> Ainsi on peut dire que, de quelque maniere que Dieu auroit [a] créé le monde, il auroit toujours esté regulier et dans un certain ordre general. Mais Dieu a choisi celuy qui est le plus parfait, c'est à dire celuy qui est en même temps le plus simple EN HYPOTHESES, et le plus riche EN PHENOMENES, comme pourroit estre une ligne de geometrie dont la construction seroit aisée et les proprietes [b] et effects seroient fort admirables et d'une grande étendue. *Je me sers de ces compa-*

Discours de métaphysique VI
(Courtesy of https://gallica.bnf.fr/)

Please note that here Leibniz uses the word *simple* for simplicity and *fort composée* for complexity, because there was not yet a word for complexity in the French language.

The *Discours* considers a graph with a finite number of points giving the behavior $f(t)$ of a physical system as a function of time t. The **complexity** is measured via the size of an equation passing through this finite set of points. AIT (algorithmic information theory) changes the context. Now one considers a finite string of bits S, the theory is a binary program P that outputs S, and the complexity of the theory P is the number of bits in P.

If there is no theory P substantially smaller than S, then S is **irreducible** or algorithmically random. Most n-bit strings S are, they require programs P of almost the same size n.

Furthermore, an **elegant program** P is the smallest one that calculates its output S, it is the best, the simplest theory for its output S.

Before continuing our analysis of this fundamental text by Leibniz, some words about the history of AIT.

Program-Size Complexity, Algorithmic Information Content, and Conceptual Complexity: Andrey Kolmogorov, Ray Solomonoff, Gregory Chaitin, c. 1965

In 1965 there was an Iron Curtain between the West and the East. There was only one computer science journal in the West, called the *Journal of the Association for Computing Machinery*. Computational complexity theory was just beginning, and most people were interested in run-time complexity. Only three of us preferred to look at program-size complexity instead of run-time complexity.

There was Andrey Kolmogorov in the Soviet Union, in Moscow, a distinguished elderly mathematician, Ray Solomonoff in Cambridge, Massachusetts, who was a friend of Marvin Minsky at MIT, and there was me, an undergraduate at the City College in New York City.

Ray was interested in machine learning and artificial intelligence, like Marvin Minsky, and in particular he was interested in the problem of induction, in how to predict the future from the past.

Kolmogorov and I were mathematicians and we both wanted to be able to define a random, structureless or patternless finite string or infinite sequence of bits. And I, I alone, thought that this must have something to do with Gödel incompleteness, as of course turned out to be the case, with a vengeance.

None of us were aware of Leibniz's highly relevant reflections in the *Discours de métaphysique.* I only discovered Leibniz much later in life, through Weyl, although a "rational reconstruction," as Imre Lakatos called it, must necessarily start with Leibniz, as we are doing here.

Returning now to Leibniz

In the above excerpt from the *Discours,* Leibniz asserts that this is the best of all possible worlds in the sense that God minimizes the bricks, the laws He uses to build the world, and simultaneously maximizes the diversity and richness of the resulting world. The beauty, simplicity and harmony (and indeed the comprehensibility) of the world are aspects of God's perfection.

This is how Leibniz shows that theism and the new mechanical philosophy, now called modern science, are actually compatible—in part because he thought that atheism would lead to chaos and social breakdown!

Choosing the Programming Language

In order to use the notion of program-size complexity as a mathematical tool, one has to decide on the programming language. One possibility is to choose a real programming language, like LISP, and to measure the size of LISP S-expressions in characters of code. This works to a certain extent.

But for the best mathematical results it is better to use somewhat artificial binary programming languages in which program-size complexity is measured in bits. Such programming languages are defined mathematically as follows: We somehow pick a particular universal machine U that can imitate any other computer C given program p by only adding a fixed number of bits of information to p, a self-delimiting prefix π_C:

$$U(\pi_C\, p) = C(p).$$

In other words, prepend π_C to p to convert a program for C into a program for U. And to make π_C self-delimiting, just double each bit and place a pair of unequal bits at the end as punctuation.

The intellectual motivation for all of this was to analyze mathematically the scientific method, to get a grasp of the complexity of a scientific theory. This was what interested Solomonoff the most. But what I wanted to do was apply these new ideas, this new toolkit, to incompleteness.

Here we go!

An elegant program is the smallest program for its output and also the best theory for its output

Actually it is better to say that an elegant program is one with the property that no smaller program written in the same language produces the same output that it does, because there may be ties. There may be several elegant programs for the same output, all with the same size.

There are only finitely many provably elegant programs!

Proof: Consider the first provably elegant program that is substantially larger in size than the software for enumerating all the theorems of your formal axiomatic theory. But this yields an even smaller program for that supposedly elegant program's output!

This software must be written in the same programming language as the elegant programs that one is considering.

In my opinion, this is the most natural approach to incompleteness. It makes incompleteness obvious, it just hits you in the face!, thus illustrating Alexandre Grothendieck's maxim that when you find the right context for a proof, the proof becomes trivial.

The Halting Problem Again

Corollary: The unsolvability of the halting problem, because if you can solve the halting problem for all programs up to a certain size, then you can determine all the elegant programs up to that size. The size can be measured in characters or in bits, depending on the programming language being considered.

Converse: determining as many elegant programs as possible

The axiom you need to know is either the $\leq N$-character program that takes longest to halt, or the N-bit string giving the number of programs $< N$ bits in size that halt.

With this information one can solve the halting problem for all programs $\leq N$ characters in size or $< N$ bits in size.

Epistemology as Information Theory

What we have presented in this chapter is actually a complete reformulation of epistemology in terms of information theory, in terms of algorithmic information. We have analyzed both the scientific method and the capabilities and limitations of formal axiomatic theories, both physics and mathematics, in terms of information. And from this information-theoretic point of view, scientific theories and mathematical theories do not look that different.

*** The Platonic World of Mathematical Ideas is Quasi-Empirical ***

As another Russian mathematician, Vladimir Arnold, put it, mathematics is just like physics, except that the experiments are cheaper! No messy wetlabs, no giant accelerators, just computer experiments!

But in both cases one seeks to unify, to simplify, to find common principles to organize ones experiences, physics experiences or math experiences, as the case may be.

In other words, theories are compressions of our experience, be it in physics or in mathematics.

For a more nuanced argument, please see the essays by Imre Lakatos and Gregory Chaitin in the collection Tymoczko, *New Directions in the Philosophy of Mathematics* cited below. See also the extensive quasi-empirical remarks by Kurt Gödel in the chapter about him.

In the next chapter, we take this train of thought to the next level. We reformulate AIT in terms of self-delimiting programs and crash into the halting probability Ω, which even refutes Leibniz's principle of sufficient reason!

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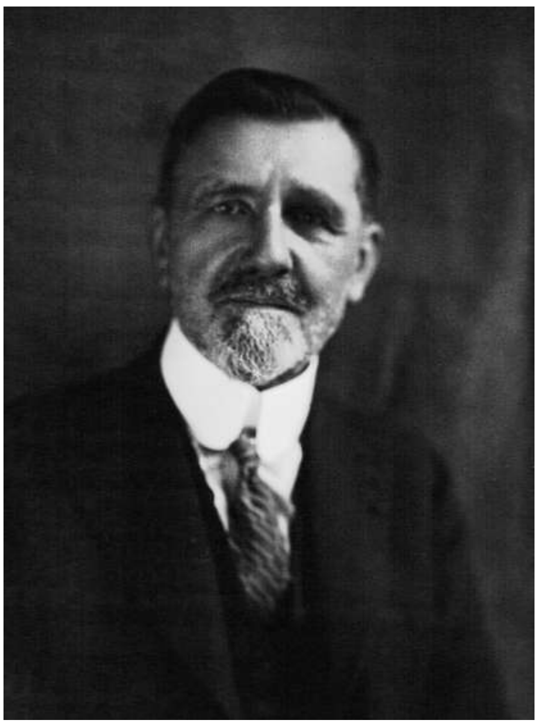
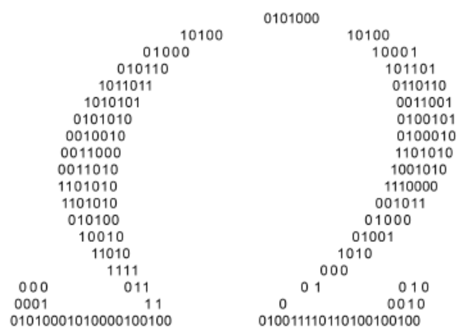
Chapter 9

Émile Borel and the Halting Probability Omega, 1974



Émile Borel



The Halting Probability

(À droite, une métaphore de l'Oméga de Chaitin par Jean-Christophe Benoist)

EL AZAR, 1935

Many years ago, in my paternal grandparents' home in Buenos Aires, I found a book with the intriguing title *El Azar*, randomness.[1] It was written by Émile Borel, a member of the first generation of mathematicians who adopted a more abstract set-theoretical

[1]Originally published in 1914 by Félix Alcan as *Le Hasard*.

point of view, and one of the creators of measure theory and modern probability theory. In 1909 he proved the following beautiful theorem:

Borel's proof that almost all reals are "normal," 1909

With probability one, a real number in the unit interval has the property that in any base b and for blocks of digits of any fixed size k, the limiting relative frequencies will exist and be equal to b^{-k}. The set of non-normal reals in the unit interval has measure zero, they are possible but infinitely unlikely.

Here an interval's measure (or probability) is its size. The entire unit interval has probability one, and a fixed initial n-bit string of bits in the base-two numeral for a real x corresponds to a subinterval of the unit interval of size 2^{-n}.

For a concise, self-contained proof, see the work by Hardy and Wright in the Literature section below.

Borel's know-it-all real number, 1927

The nth digit answers the nth yes/no question in French![2]

Change this to: The nth bit tells us whether the nth computer program ever halts!

Then observe that n instances of the halting problem are only $\log_2 n$ bits of information, because you only need to know how many halt to find out which ones halt.

This leads us to the halting probability Ω, whose first n bits tells us for each program up to n bits in size, whether or not it halts. And Ω is the most compressed possible way to do this, it is totally irredundant, totally incompressible. From this it is possible to prove that Ω is a Borel normal real number.[3]

After this introduction involving Borel, let's define the halting probability Ω more carefully.

Picking a New Universal Computer U

Recall that in the previous chapter we stipulated that

$$U(\pi_C\, p) = C(p)$$

Now not only π_C must be self-delimiting, p must also be self-delimiting.

To make an arbitrary n-bit string self-delimiting, put in front a prefix giving n in base-two, with each bit repeated and two unequal bits at the end. The result is

$$n + 2\log_2 n + 2$$

bits long. You can iterate this, resulting in

$$n + \log_2 n + 2\log_2 \log_2 n + 2$$

[2]See Borel, *Leçons sur la théorie de fonctions.*

[3]See *Cambridge Tracts in Theoretical Computer Science, Vol. 1*, 1987.

bits, or

$$n + \log_2 n + \log_2 \log_2 n + 2 \log_2 \log_2 \log_2 n + 2$$

bits, etc. This is not quite right because the base-two numeral for n is not exactly $\log_2 n$ bits long, but it gives the general idea.

Does an N-bit String Have Program-Size Complexity N or $N + \log_2 N$?

More precisely, the best way to make an N-bit string self-delimiting is to precede it by the smallest possible self-delimiting program to calculate N, which is by definition $H(N)$ bits long. More generally, $H(X)$ denotes the size in bits of the smallest self-delimiting program to calculate the digital object X, the *algorithmic information content* of X.

So most N-bit strings require programs of about this number of bits:

$$N + H(N).$$

These are the irreducible N-bit strings. As above, $H(N)$ can be approximated by $\log_2 N$ plus additional logarithmic terms, but is sometimes, rarely, much, much smaller, if N is of a special form, such as a power of 2, or a factorial number, 2^n or $n!$.

That one should rework AIT in this manner, basing it on self-delimiting programs, was realized by Gregory Chaitin and by Leonid Levin, c. 1974. The resulting theory of program-size complexity is extremely elegant, featuring the following *subadditivity* property

$$H(X, Y) \leq H(X) + H(Y) + c,$$

which says that it is possible to get a program to calculate the ordered pair X and Y by using c bits to stitch together the individual programs to calculate X and Y.

More precisely, this inequality says that (the algorithmic information content of the ordered pair X and Y) is bounded by (the algorithmic information content of X) plus (the algorithmic information content of Y) plus a constant c.

This may be restated more gracefully as follows: The *joint information* of X and Y is bounded by the sum of the individual information contents of X and Y plus a constant.

Then one can define the *mutual information* between X and Y to be

$$H(X) + H(Y) - H(X, Y),$$

and the *relative information* of X given Y to be

$$H(X, Y) - H(Y).$$

Furthermore, here is the equation for the halting probability Omega:

$$0 \; < \; \Omega \; = \sum_{U(p) \text{ halts}} 2^{-(\text{size in bits of } p)} \; < \; 1.$$

In other words, Ω is the total probability of all the programs p that halt when run on U, in which an N-bit program has probability 2^{-N}. This converges, and to a sum < 1, only because we have switched to self-delimiting programs. Otherwise this sum would diverge to ∞ and we could not define the halting probability.

Complexity dips in infinite sequences

Consider an infinite sequence of independent tosses of a fair coin. Can we define an algorithmically random or irreducible infinite sequence of bits to be one for which the complexity of its initial N-bit segments stays as high as possible, namely, near to $N + H(N)$? No! Why is that? Well, it's because there must be complexity dips due to runs of identical bits as dictated by the law of the iterated logarithm in William Feller's classic *An Introduction to Probability Theory and Its Applications, Vol. 1.* By the way, the law of the iterated logarithm is originally due to Aleksandr Khinchin in 1924.

Now we get to the mathematical climax of this chapter, which is that there are in fact three different definitions of an irreducible infinite sequence of bits which look very different but that in fact turn out to be equivalent, always a good sign. Here they are:

Three Very Different Definitions of a Random Infinite Sequence/Real Number ρ

- *Gregory Chaitin:* There is a constant c such that for every N, the first N bits of the sequence ρ have program-size complexity greater than $N - c$.

- *Per Martin-Löf:* The real number ρ is not contained in any set of real numbers (in the unit interval with uniform measure) of constructive measure zero, i.e., with arbitrarily small computably enumerable covering, a computably enumerable covering that is uniformly computable given the arbitrarily small size $\epsilon = 2^{-N}$ of the covering that is desired.

  Note that the union of all sets of reals of constructive measure zero is also of measure zero, since there are only a countable infinity of such sets.[4] So almost all reals in the unit interval are Martin-Löf random, with probability one.

- *Robert M. Solovay:* His definition is similar to Martin-Löf's but more convenient mathematically since it is closely related to the Borel-Cantelli lemma as presented in William Feller's *An Introduction to Probability Theory and Its Applications, Vol. 1.* This lemma was used by Borel to show that almost all real numbers in the unit interval are normal, i.e., with probability one.

  Borel–Cantelli lemma: If the sum of the probabilities of an infinite sequence of events $A_n, n = 0, 1, 2, \ldots$ is finite, that is, converges,

  $$\sum_n \Pr(A_n) < \infty$$

  then the probability that infinitely many of them occur is 0. Furthermore these events do not have to be independent. The events A_n are in (are subsets of) the unit interval of real numbers x $\{0 \le x \le 1\}$ with uniform measure.

  Solovay criterion: A real number ρ is random if it is never contained in infinitely many A_n that are constructively defined as a function of n and for which $\sum_n \Pr(A_n)$ converges.

[4] A set is said to be a countable infinity if it can be put in a one-to-one correspondence with $\{1, 2, 3, \ldots\}$.

Ω satisfies all three of these definitions of randomness.[5] Therefore, for example, Ω is Borel normal in every possible base for blocks of digits of any fixed finite size. The limiting relative frequencies always exist and are identical, no matter the base/radix in which the numerical value of Ω is written.

Infinite Irreducible Complexity in Pure Mathematics

We have spent a great deal of time in this chapter on technical mathematics. But let's now turn to philosophy and to epistemology.

Consider whether each individual bit in the base-two expansion of the numerical value of

$$\Omega = .0111001010\ldots$$

is a 0 or a 1.

An N-bit formal axiomatic theory can determine at most $N + c$ bits of the base-two numerical value of Ω. When we say that something is an N-bit theory, we mean that the software for enumerating all the theorems of the formal axiomatic theory has N bits. But you have to remember that in this chapter all programs have to be self-delimiting, even ones that go on forever.

The numerical values of the individual base-two bits of Ω are an infinite sequence of assertions that are true for no reason!

Thus the bits of Ω violate Leibniz's principle of sufficient reason, which states that if something is true it must be true for a reason. In mathematics, of course, the reason is called a proof, and the job of the mathematician is to find the reason.

Please note that anything can be proven by adding it, or something equally complex, as a new axiom. So this doesn't count as a proof. But if you want to determine the numerical values of the individual base-two bits of Ω, this is the only way that you can do it.

Necessary truths versus Contingent truths

The bits of Ω are a perfect simulation in pure mathematics, where all truths are necessary, of contingent truths, more precisely, of infinitely many independent tosses of a fair coin.

Does God play dice in pure mathematics?

Einstein asserted that God doesn't play dice in the physical universe, that everything is deterministic, predictable in principle, at least to Laplace's famous demon. Nevertheless, it appears that God does play dice in pure mathematics. The bits of Ω show

[5] There are two books proving this in the Literature section below.

that this is so. Is this a nightmare for the rational mind? Not really! It actually sends us off in the direction of biology.

*** The Platonic World of Mathematical Ideas is Biological ***

Ω shows that mathematics is more biological than biology itself. Biology has immense but finite complexity, but mathematics provably has infinite complexity!

The complexity of the field of biology is illustrated by the size of this book: *Molecular Biology of the Cell, 7th Ed.*, with seven co-authors, published in 2022, 1552 pages! Or look at the size of the human genome: 3.2 billion base pairs. Whereas one can put the fundamental equations of mathematical physics on (both sides of) a college student's T-shirt. And fields of mathematics usually present their axioms in a few pages.

Based on this clue, in the next chapter we shall take a look at creativity in mathematics and in biology. But before doing that, I want to tell you a little bit more about Ω, some rather technical material which however also has some philosophical significance.

An exponential diophantine equation for the bits of the halting probability Omega

Yuri Matijasevic and James P. Jones have shown how to translate register machine programs into exponential diophantine equations, diophantine equations with variable exponents, using a beautiful theorem of Édouard Lucas regarding the even/odd parity of binomial coefficients. The resulting equation has many variables and has only one solution if the register machine program halts and no solutions if the register machine program fails to halt. Using their work it is easy to construct a (large) exponential diophantine equation with a parameter k which has infinitely many solutions if the kth bit of Ω is a 1 and which has only finitely many solutions if the kth bit of Ω is a 0.

In my 1987 Cambridge University Press monograph (see below), this was done for a specific halting probability Ω that can be calculated very, very slowly in the limit from below[6] using a LISP program written in a toy version of LISP for which an interpreter was programmed on a register machine.

Therefore, the halting problem and the halting probability can both be expressed as arithmetical statements. They are not just about theoretical computer science.

Choosing a specific universal computer U for algorithmic information theory

I did this in 1998 using yet another specially designed dialect of LISP, for which I wrote an interpreter in C. Then it is possible to run programs on U and to show that, for

[6]There is of course no regulator of convergence or this Ω number would be a computable real number, which is not at all the case.

example, an N-bit theory can determine at most $N + 15328$ bits of Ω. Without picking a specific U, and furthermore one that is easy to program, one is only able to show that an N-bit theory can determine at most $N + c$ bits of Ω, with absolutely no idea how big c may actually be.

Literature

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2. Émile Borel, *Leçons sur la théorie de fonctiones (Principes de la théorie des ensembles en vue des applications à la théorie des fonctions)*, Gabay, 2003.

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4. William Feller, *An Introduction to Probability Theory and Its Applications, Vol. 1*, Wiley, 1968. [Where the author of this book learned his probability theory.]

5. Gregory Chaitin, *Algorithmic Information Theory, Cambridge Tracts in Theoretical Computer Science, Vol. 1*, Cambridge University Press, 1987. [Proves that Ω is Borel normal and that the three definitions of a random real are equivalent.]

6. Gregory Chaitin, *The Limits of Mathematics: A Course on Information Theory and the Limits of Formal Reasoning*, Springer-Verlag, 1998. [Proves that an N-bit theory can determine at most $N + 15328$ bits of Ω.]

7. Gregory Chaitin, *Exploring Randomness*, Springer-Verlag, 2001. [Also proves that the three definitions of a random real turn out to be equivalent.]

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Video

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Chapter 10

Creativity in Mathematics and in Biology, 2012

> "A mathematician who is not also something of a poet will never be a complete mathematician."—Karl Weierstrass

LEONHARD EULER.

Srinivasa Ramanujan

Mathematical creativity

One paper by Euler every week on beautiful new mathematics, in every possible field, even after he went completely blind! Volume after volume of his collected works, his *Opera Omnia*. What could the source of all this creativity be?!

And how about Ramanujan, who affirmed that "An equation for me has no meaning, unless it expresses a thought of God." And that a goddess wrote equations on his tongue whilst he slept.

Ramanujan was G. H. Hardy's greatest discovery. One day at Cambridge University Hardy received a letter from an unknown in India containing amazing formulas, spectacular infinite continued fraction expansions and startling infinite series expansions. "Is an imposter of genius more likely than a mathematician of genius?," Hardy asked himself. Clearly, no! Such was Hardy's analytic mind at work. It was World War I, Hardy's friend Bertrand Russell had been dismissed from Cambridge because he was a pacifist, and Hardy's best students were dying in the trenches. Ramanujan became Hardy's great project, his solace amongst the slaughter, because Ramanujan had no idea what a proof was, he worked entirely by some kind of intuition and feeling for form, by inspiration, no less!

So Hardy tried to teach Ramanujan what a proof was, and together they wrote some wonderful papers. As C. P. Snow says, for once, virtue was rewarded. But it was fatal for Ramanujan, who was away from his wife, could barely survive in medieval Cambridge without heat or hot water, and, during wartime, could not obtain the fruits and vegetables that he as a vegetarian required. Ramanujan went back to India to die leaving notebooks full of unfinished work.

Biological creativity

Such is the mystery of mathematical creativity at its finest. Now let us turn to biological creativity as explicated by Charles Darwin, in particular in his youthful work *The Voyage of the Beagle*, a story of adventure and insight. Darwin, a naturalist, was greatly impressed by the variety of forms he encountered in his voyage around the world in the Beagle, and after reading Malthus, postulated that random variations supplemented by selection, as exercised by his gentlemen friends who bred prize roses or champion horses, could eventually lead to the evolution of new species.

Is randomness really enough of a creative force to accomplish this? That is the question. Perhaps mathematics can shed some light on this. After all, in mathematics we do know something about creativity. Gödel's incompleteness theorem is usually viewed pessimistically, but it can also be viewed as showing that math offers unlimited scope for imagination and creativity, that it is not a closed system.

That is what metabiology attempts to accomplish, to transform incompleteness and uncomputability into a toy model of biological evolution. The key idea, as I stated in the chapter on von Neumann, is to study the random evolution of computer programs, artificial software, instead of the random evolution of DNA, which is natural software.

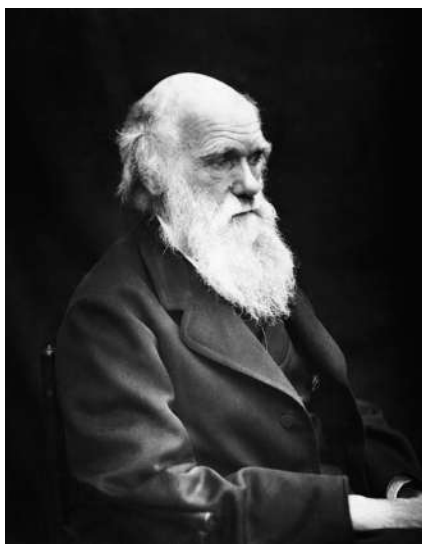
I can actually prove a few little theorems about how such a system can evolve, which suggests to me that this might possibly be a fruitful area of research.

I should say a little more about my toy model. Mathematically, it amounts to a hill-climbing random walk in software space, because mutations are ignored unless they can increase the fitness. Furthermore in my model, evolution is not driven by adaptation to the environment. It is driven by intrinsic, endogenous creativity. My software organisms are mathematicians who wish to become better, to climb to higher and higher constructive ordinals, as it were. So there is no limit to how far they can go.

Recall Lee Van Valen's *Red Queen principle*, which states that evolution is intrinsic, not driven by changes in the environment as is usually thought. You have to run as fast as you can to stay in the same place, because everyone else is also doing so!

Another amusing aspect of the model is that it turns out that localized mutations such as indels and snips make for an extremely ugly toy model, at least from a mathematical point of view. Instead I had to postulate global algorithmic mutations, which turns out to correspond in some ways to the fact that the Extended Evolutionary Synthesis has also concluded that richer mutational mechanisms are needed to account for the bursts of creativity that we see in the fossil record, such as the Cambrian explosion, or with punctuated equilibrium.

Perhaps this line of research is too conceptual, too theoretical, too abstract for our pragmatic age. But if I were a little bit younger I would strive to continue in this direction. Fortunately my wife, epistemologist and philosopher of science Virginia Chaitin, and my former Rio de Janeiro student Felipe Abrahão, are doing so. Furthermore Hector Zenil, on whose Ph.D. committee I was happy to sit some years ago, has been exploring practical applications of related ideas. Indeed, he has built a startup called Oxford Immune Algorithmics, and Felipe is heading to the U.K. to participate. So perhaps not all is lost. I look forward to hearing what they accomplish.



Charles Darwin

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7. Virginia Chaitin, "Metabiology, Interdisciplinarity and the Human Self-Image," in Wupperluri, Doria, *Unravelling Complexity: The Life and Work of Gregory Chaitin*, World Scientific, 2020.

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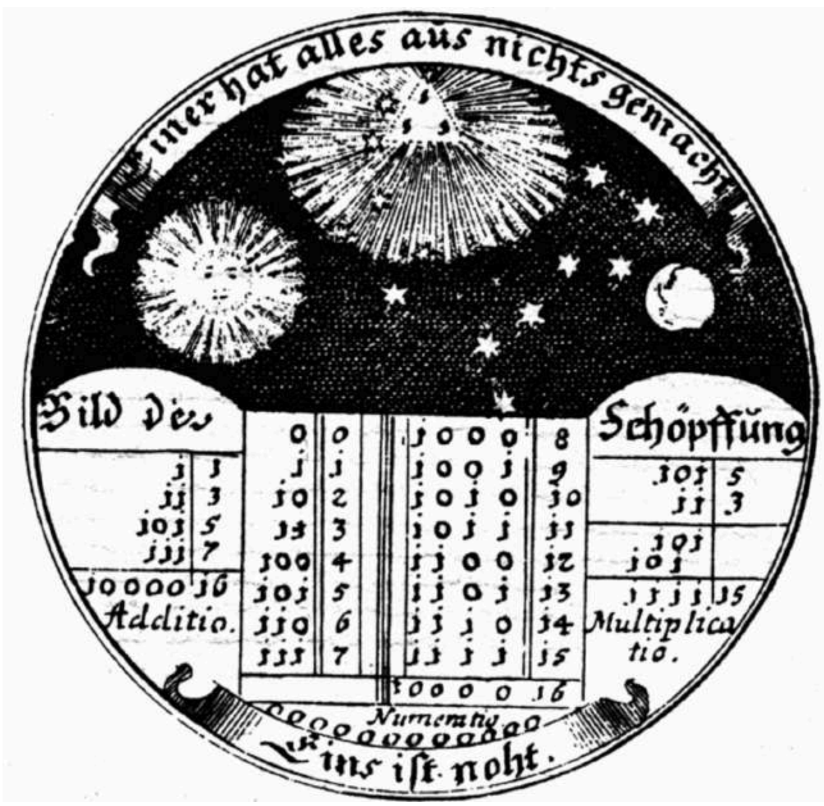
Chapter 11

Against Materialism: Information and Consciousness

"As God cogitates and calculates, so the world is made."—Leibniz



Digital Philosophy

According to Tobias Dantzig in his classic, *Number, the Language of Science*:

> It is the mystic elegance of the binary system that made Leibnitz exclaim: *Omnibus ex nihil ducendis sufficit unum.* (One suffices to derive all out of nothing.) Says Laplace:

> "Leibnitz saw in his binary arithmetic the image of Creation ... He imagined that Unity represented God, and Zero the void; that the Supreme Being drew all beings from the void, just as unity and zero express all numbers in his system of numeration. This conception was so pleasing to Leibnitz that he communicated it to the Jesuit, Grimaldi, president of the Chinese tribunal for mathematics, in the hope that this emblem of creation would convert the Emperor of China, who was very fond of the sciences. I mention this merely to show how the prejudices of childhood may cloud the vision even of the greatest men!"

And MIT Press has recently published an important addition to the Leibniz literature, Strickland and Lewis, *Leibniz on Binary*, with much new information about Leibniz's vision that the entire universe may be built out of information, out of 0s and 1s, which we now call *digital philosophy*:

- Pythagoras, Plato: **All is number, God is a mathematician!**
- Leibniz, Digital Philosophy: **All is algorithm, God is a programmer!**

This vision is a fitting culmination to all the material on computation and information throughout this book. And it also ties in with David Chalmers' speculation in his *The Conscious Mind: In Search of a Fundamental Theory* that information theory may provide the basis for a theory of consciousness.

Indeed, how about a world in which mind and information are primary, and matter is an epiphenomenon? This is a panpsychist scenario, in which different minds or different monads have differing degrees of perception, as in the Leibniz *Monadology*. As my wife, philosopher Virginia Chaitin, points out, perhaps this is also analogous to the different degrees of perception made possible by Turing oracles O_α.

But is it possible to identify the fundamental nature of reality by pure thought alone? What does Nature have to say? Well, Nature speaks with many voices.

The history of physics, as presented in Einstein and Infeld, *The Evolution of Physics*, has been marching away from materialism and in the direction of idealism, in my opinion. From the mechanical philosophy as practiced by Galileo, Newton already introduces non-materialistic action at a distance, which only gets worse with Maxwell's electromagnetic fields and Einstein's field theory of gravity, which postulate infinitesimal propagation rather than action at a distance.

The big break is with quantum mechanics, which has been with us for a century, and which in my opinion is clearly an idealistic theory. For the Schrödinger equation does not deal with electrons, it deals with the Schrödinger ψ function, a probability field with phase, which contains our knowledge about the electron.

Perhaps this is even clearer with quantum information, quantum computation and qubits, which share an obvious analogy with algorithmic information theory and its classical bits.

I must, however, express my discontent with quantum mechanics, which as I said has already been with us, its fundamental assumptions unchallenged, for an entire century. I attribute this more to the sociology of science, i.e., to what Einstein would refer to as "the herd instinct" than to the, in my opinion, mistaken view that quantum mechanics is the final theory. Indeed, there are serious observational anomalies that our current physics does not seem able to explain: the dark matter, the energy source of the solar corona, and ball lightning.

Amazingly enough, Randell Mills has shed some light on these matters. Building on the work of one of his professors at MIT, Hermann Haus, Mills has developed a classical theory of the hydrogen atom, with the consequence that according to Mills there are states of the hydrogen atom below the conventional ground state according to quantum mechanics. In other words, according to Mills' theory, there are stable hydrogen atoms in which the electron is orbiting closer to the proton than is permitted in conventional quantum mechanics. These he calls *hydrinos*, and he believes that they are the infamous dark matter and can explain the unaccountably high temperature of the solar corona.

Mills and his colleagues have accumulated a substantial amount of experimental evidence for hydrinos that has been published in reputable journals, though naturally there is still considerable room for skepticism regarding the theory that predicted them.

Furthermore, he has built prototype devices for creating hydrinos that he puts forward as, potentially, important new sources of green energy. Please take a look at his website at https://brilliantlightpower.com or at the book by by Brett Holverstott, *Randell Mills and the Search for Hydrino Energy*.

Please stay tuned for further developments!

Changing topic, I should like to tell you about a remarkable development, Stephen Wolfram's metaphysics of the *Ruliad*, the entangled sum of the time evolution of all possible discrete, algorithmic laws of physics and all possible mathematical formal axiomatic theories, hence the title of one of his many recent books, *Metamathematics: Foundations & Physicalization*. This is a breathtaking new idea, and constitutes an approach to metaphysics that is completely orthogonal to the work presented in this book. Highly recommended!

We have reached the end of our journey, the end of the story of my personal quest to understand the incompleteness phenomenon discovered by Kurt Gödel. But the human spirit will continue to ponder and question, hopefully forever. Perhaps you, dear reader, can illuminate these questions with new viewpoints and new insights? Or perhaps this will be achieved by the artificial super-intelligences that have now appeared on the horizon?

Literature

1. Tobias Dantzig, *Number: The Language of Science*, Macmillan, 1930. [Contains Laplace's criticism of digital philosophy as envisioned by Leibniz.]

2. Lloyd Strickland, Harry R. Lewis, *Leibniz on Binary: The Invention of Computer Arithmetic*, MIT Press, 2022. [About Leibniz on digital philosophy, not just about computer hardware. Look at the cover!]

3. David J. Chalmers, *The Conscious Mind: In Search of a Fundamental Theory*, Oxford University Press, 1996. [With an entire chapter on information theory.]

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7. Stephen Wolfram, *How to Think Computationally about AI, the Universe and Everything*, https://writings.stephenwolfram.com/2023/10/how-to-think-computationally-about-ai-the-universe-and-everything/

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3. Gregory Chaitin, "Leibniz on complexity," Singapore, 2016: https://www.youtube.com/watch?v=naZHlXgeXkc

Chapter 12

Transhumanism or Posthumanism?

In this book we have, so to speak, been concerned with Algorithms 1.0, that are written in programming languages, not with Algorithms 2.0, that are embodied in neural nets. This is indeed a momentous paradigm shift.

Marvin Minsky at MIT, who liked to provoke people, used to declare that we are a carbon-based life-form that is creating a silicon-based life-form to replace us. The spectacular recent success of neural net artificial general intelligence (AGI) has made Minsky's provocation into a real possibility.

Elon Musk has declared that the goal of his new company xAI is "to understand the true nature of the universe," a laudable goal, and excellent for recruiting. But if he achieves this goal, will we be able to understand what his artificial super-intelligence has achieved?

As my wife, philosopher Virginia Chaitin, reminds me, Giorgio Colli in his *La nascita della filosofia* states that the problem with trying to understand the prophecies of the gods, even when they do not wish to fool us, is that human language cannot express thoughts at the level of perception of the gods. It would be like lecturing on Goethe's *Faust* to an ant in the garden.

And we may have the same difficulty with an insight from an AGI that understands 80 human languages and their corresponding cultural contexts. As Virginia points out, this is analogous to a mind containing the Turing oracle O_α trying to understand a thought realized by a mind with an oracle O_β with $\alpha < \beta$.

Referring again to Musk, perhaps a high-bandwidth Neuralink connection between humans and AGIs will result in a symbiosis, rather than a replacement?

As it says in Mark Twain's *Huckleberry Finn*, you pays your money and you takes your choice.

Literature

1. Robert J. Marks II, *Non-Computable You: What You Do That Artificial Intelligence Never Will*, Discovery Institute Press, 2022.

2. Mo Gawdat, *Scary Smart: The Future of Artificial Intelligence and How You Can Save Our World*, Pan Macmillan, 2021.

Appendix A

Cum Deus calculat...fit mundus

> "As God cogitates and calculates, so the world is made."—Leibniz

Stephen Wolfram was curious about the source of this Leibniz quotation that heads Chapter 8 and Chapter 11. Here we report what he found. First of all, what did Leibniz actually say? The original Latin is this:

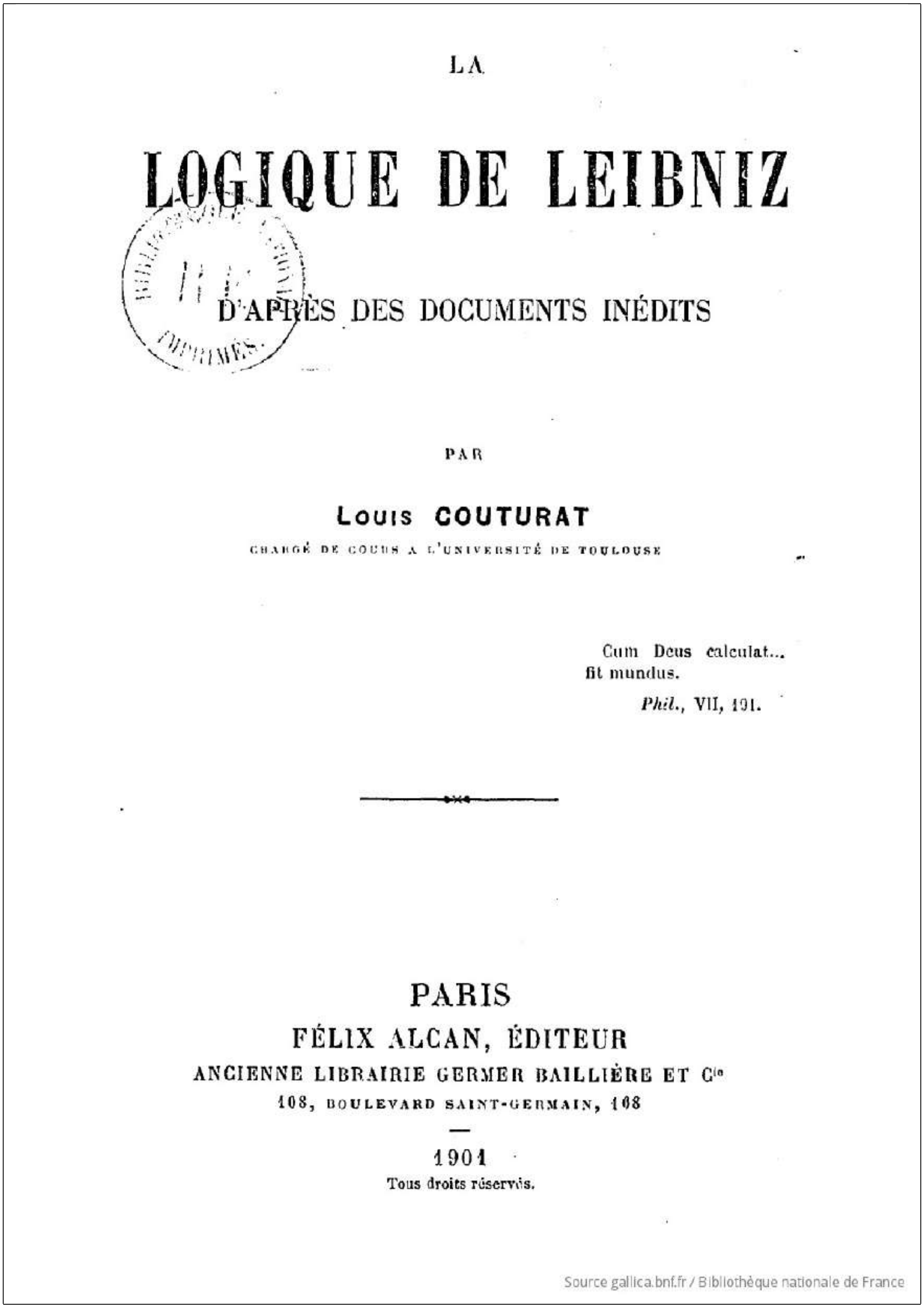
> "*Cum Deus calculat et cogitationem exercet, fit mundus.*"—Leibniz

However, the version that the author of this book first found, in a text by Heidegger, is abbreviated:

> "What does Leibniz say about God in regard to the universe? In 1677 (at the age of thirty-one) Leibniz wrote a dialogue on the *Lingua rationalis*, that is, on calculus, which is the sort of reckoning that is in the position of giving a full accounting of the relations between word, sign, and thing—and thus for everything that is. In this dialogue and in other essays, Leibniz had anticipated the fundamentals not only for what today are used as thinking machines, but even more, of what determines their manner of thinking. In a hand-written marginal note to this dialogue Leibniz remarks: *Cum Deus calculat fit mundus*. When God reckons, a world comes to be. All that is needed is a ready and willing glance into our atomic age in order to see that if God is dead, as Nietzsche says, the calculated world still remains and everywhere includes humans in its reckoning inasmuch as it reckons up everything to the *principium rationis*."—Heidegger, *The Principle of Reason*

What happened? What happened is that this remark by Leibniz became known because Louis Couturat placed it on the cover of his influential 1901 book *La Logique de Leibniz*. But as you can see below, where we reproduce that cover, Couturat abbreviated Leibniz's Latin to *Cum Deus calculat...fit mundus*, and then Heidegger removed the ellipsis.

At any rate, the original source is a marginal note in a short *Dialogus* dated 1677. You must remember that the immense Leibniz *Nachlass* conceals many treasures. Little was actually published during his lifetime, and there is no definitive statement of his views, which were in a constant state of evolution.



LA

LOGIQUE DE LEIBNIZ

D'APRÈS DES DOCUMENTS INÉDITS

PAR

Louis COUTURAT

CHARGÉ DE COURS A L'UNIVERSITÉ DE TOULOUSE

Cum Deus calculat... fit mundus.

Phil., VII, 191.

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Couturat, *La Logique de Leibniz*, 1901
(Courtesy of https://gallica.bnf.fr/)

Appendix B

Interviews

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2. "Consistent Systems are 'Boring,' Paradoxes are Fertile," Gödelian Letters by Sami Al-Suwailem, 2023: https://medium.com/godelian-letters/interview-with-gregory-chaitin-b4e63460c94b
3. "Digital Philosophy and Omega," Just Nowhere with Samuel Zinner, 2022: https://www.youtube.com/watch?v=ht3jxGPhHo8
4. "A Conversation between Gregory Chaitin and Stephen Wolfram at the Wolfram Summer School," 2021: https://www.youtube.com/watch?v=d8MWRkS1pek
5. "A Conversation between Gregory Chaitin and Stephen Wolfram, Part 2," 2021: https://www.youtube.com/watch?v=Mtt6PWt0Kcg
6. "Randomness, Information Theory, and the Unknowable" with Robert J. Marks, 2021: https://mindmatters.ai/podcast/ep167/
7. "Why Are We Here?" with Ard Lewis and David Malone, 2017: https://www.whyarewehere.tv/people/gregory-chaitin/
8. "Against Method" by Karol Jałochowski, 2014: https://vimeo.com/171062577, https://www.imdb.com/title/tt5127090/, https://www.youtube.com/watch?v=uEtqJSRz3mE [Also featuring Virginia Chaitin.]

About the Author

Please see "The Enigmatic World of Gregory Chaitin" at https://www.youtube.com/watch?v=z4FWfr6tf0A. *Erratum*: Chaitin was awarded the Leibniz Medal in 2007, not in 2012.

Goya—El sueño de la razón produce monstruos